CAPS MATCH 2026

Solutions Booklet

Each problem is worth 7 points.

Contents

Preamble

This booklet contains the official problems and solutions for the CAPS MATCH 2026. The problem selection team gratefully acknowledges problem submissions from the following countries:

- Austria (5 problems)
- Czechia (5 problems)
- Poland (4 problems)
- Ukraine (6 problems)

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Problem 1

Problem 1. In a sequence $a_1, a_2, \ldots, a_{100}$ of 100 mutually different real numbers, we say that an index $i \in \{2, 3, \ldots, 99\}$ is *good* if $a_1 + \cdots + a_{i-1} = a_{i+1} + \cdots + a_{100}$. What is the largest possible number of good indices?

Solution 1

Answer: 66.

1. Proof of the upper bound: We claim that there can be at most 2 consecutive good indices. Once shown, the claim implies that from among the $98 = 3 \cdot 32 + 2$ candidate indices $a_2, \ldots, a_{99}$ at most $2 \cdot 32 + 2 = 66$ can be good. To show the claim, suppose indices i and $i+1$ are both good. Then

$$\begin{aligned} a_1 + \cdots + a_{i-1} \qquad &= a_{i+1} + a_{i+2} + \cdots + a_n \quad \text{and} \\ a_1 + \cdots + a_{i-1} + a_i &= \qquad\quad a_{i+2} + \cdots + a_n. \end{aligned}$$

   Subtracting the two relations, we get $a_{i+1} = -a_i$. If a_{i+2} were also good, we would similarly get $a_{i+2} = -a_{i+1} = a_i$, a contradiction with all numbers being distinct, so the claim is proved.

2. Proof of the lower bound: By the above claim, in order for two consecutive indices i and $i+1$ to both be good, we need $a_i + a_{i+1} = 0$. By a similar argument, in order for both j and $j+2$ to be good we need $a_j + 2a_{j+1} + a_{j+2} = 0$. Motivated by this, consider the sequence

$$33, 101, -101,\ 102,\ -103, 103,\ -104,\ 105, -105,\ 106,\ -107, 107,\ -108, \ldots, 165, -165,\ 166.$$

   Then $a_3 + \cdots + a_{100} = -101 + 102 - 104 + 106 - 108 + \cdots - 164 + 166 = (-1+2) + 16 \cdot 2 = 33 = a_1$, so index $i = 2$ is good. Since $a_2 + a_3 = 0$, index 3 is good too. Since $a_3 + 2a_4 + a_5 = 0$, index 5 is also good. By trivial induction, all indices $i \equiv 0, 2 \pmod 3$ are good.

Problem 2

Problem 2. A positive integer k is called *divisive* if there exist only finitely many quadruples (x, y, z, w) of positive integers satisfying

$$x^2 + y^2 + z^2 + w^2 = xyzw \qquad \text{and} \qquad k \nmid x^2 + y^2 + z^2 + w^2.$$

Find all divisive positive integers.

Solution 1

We will show that exactly the positive divisors of 16 are divisive, i. e., $\{1, 2, 4, 8, 16\}$ is the set of all divisive positive integers.

For the rest of the proof, call a quadruple (x, y, z, w) *funky* if it consists of four positive integers satisfying

$$x^2 + y^2 + z^2 + w^2 = xyzw. \tag{1}$$

Then $k \in \mathbb{Z}_{>0}$ is divisive if $k \nmid xyzw$ for only finitely many funky quadruples.

Claim 1. All components of a funky quadruple have to be even.

Assume first, all of them were odd. Then the left hand side of (1) is even, while the right hand side is odd, impossible. Otherwise $xyzw$ is even, which implies that either exactly two or all four terms on the left hand side of (1) must also be even. If exactly two of x, y, z, and w were even, the other two were odd and their squares congruent to 1 modulo 4. Considering equation (1) modulo 4 would lead to $0 + 0 + 1 + 1 \equiv x^2 + y^2 + z^2 + w^2 = xyzw \equiv 0 \pmod 4$, again a contradiction. This concludes the proof of Claim 1.

It follows that $2^4 = 16 \mid xyzw$ for every funky quadruple (x, y, z, w), so all that remains to show that the divisors of 16 are all divisive, is proving that infinitely many funky quadruples exists. To this end, note that (2,2,2,2) is funky and perform Vièta jumping with two of the components, that is, define a sequence of pairs of positive integers by $(x_0, y_0) := (2, 2)$ and $(x_{n+1}, y_{n+1}) = (4x_n - y_n, x_n)$ for integers $n \geq 0$. Since $x_0 = y_0 = 2$, we get inductively $x_n \geq y_n \geq 2$ and $x_{n+1} + y_{n+1} = 5x_n - y_n \geq 4x_n > 2x_n \geq x_n + y_n$ for all $n \geq 0$, so the sequence $(x_n, y_n)_{n\geq 0}$ contains infinitely many pairs of positive integers.

Claim 2. $(x_n, y_n, 2, 2)$ is funky for all $n \geq 0$. The inductive proof of Claim 2 with its base case valid by construction consists of the inductive step

$$\begin{aligned} 2^2 + 2^2 + x_{n+1}^2 + y_{n+1}^2 - 2 \cdot 2 \cdot x_{n+1}y_{n+1} &= 4 + (4x_n - y_n)^2 + x_n^2 - 4(4x_n - y_n)x_n \\ &= 2^2 + 2^2 + x_n^2 + y_n^2 - 4x_ny_n = 0, \end{aligned}$$

where the last equality is the induction hypothesis.

By Claim 2, there are infinitely many funky quadruples and all positive divisors of 16 are indeed divisive integers. We proceed to show that every divisive positive integer is a divisor of 16.

Observe first that $x_0 = y_0 = 2 \equiv 2 \pmod 4$ and thus $x_{n+1} = 4x_n - y_n \equiv -y_n \equiv 2$ as well as $y_{n+1} = x_n \equiv 2 \pmod 4$ inductively for all $n \geq 0$, so none of the components of the funky quadruple $(x_n, y_n, 2, 2)$ are divisible by 4 and therefore $32 \nmid x_ny_n \cdot 2 \cdot 2$ for all $n \geq 0$ which proves that multiples of 32 are not divisive.

Assume finally that there exists some odd prime p dividing a divisive integer $k \in \mathbb{Z}_{>0}$.

Claim 3. There exists a positive integer t such that $x_{n+t} \equiv x_n \pmod{p}$ as well as $y_{n+t} \equiv y_n \pmod{p}$ hold for all integers $n \geq 0$.

By the pigeonhole principle (only p^2 pairs of residue classes modulo p exist, and there are infinitely many of the sequence $(x_n, y_n)_{n\geq 0}$), there exist integers $k > \ell > 0$ with $(x_k, y_k) \equiv (x_\ell, y_\ell) \pmod{p}$. But as the recursion is reversible, we also infer

$$(x_{k-1}, y_{k-1}) = (y_k, 4y_k - x_k) \equiv (y_\ell, 4y_\ell - x_\ell) = (x_{\ell-1}, y_{\ell-1}) \pmod{p}.$$

Continuing this way, we arrive at $(x_{k-\ell}, y_{k-\ell}) \equiv (x_0, y_0) \pmod{p}$ and setting $t := k - \ell$ yields $x_t \equiv x_0$ and $y_t \equiv y_0 \pmod{p}$, so that Claim 3 now follows easily by induction (as residue classes respect addition and multiplication).

Finally, using the period t from Claim 3 implies that $x_{mt} \equiv x_0 \equiv 2 \not\equiv 0 \pmod{p}$ and $y_{mt} \not\equiv 0 \pmod{p}$ for all integers $m \geq 0$, so that $(x_{mt}, y_{mt}, 2, 2)_{m\geq 0}$ is a sequence of funky quadruples with $p \nmid x_{mt} \cdot y_{mt} \cdot 2 \cdot 2$ and thus $k \nmid x_{mt} \cdot y_{mt} \cdot 2 \cdot 2$, contradicting that k is divisive. Therefore, only powers of 2 which are not multiples of 32 can be divisive. We have already checked above that the positive divisors of 16 are indeed divisive.

Solution 2 (Variations on necessity)

We prove two more variations on showing necessity.

Claim. In any funky quadruple, none of the components are divisible by 4.

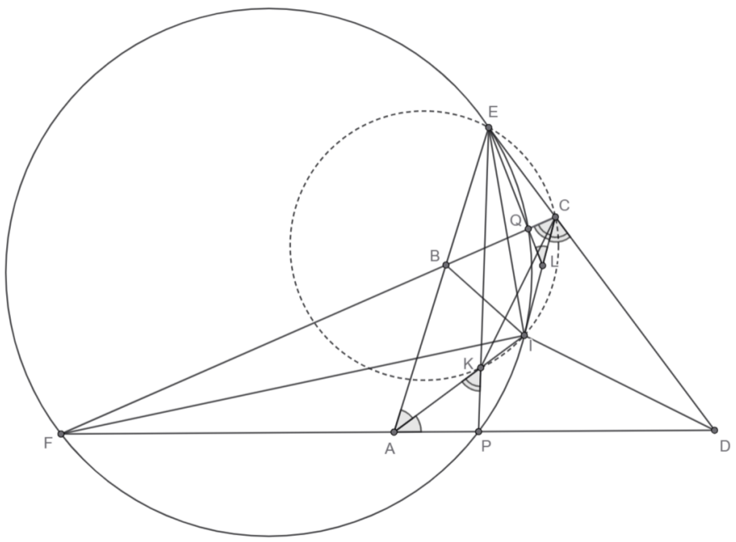
As we already know from the first part (in the first solution), all components of a funky quadruple (x, y, z, w) are even. Let w be a component for which the multiplicity of 2 is minimal, say $e \in \mathbb{Z}_{>0}$. Writing $(x, y, z, w) = 2^e(X, Y, Z, W)$ for positive integers X, Y, Z, and W with W odd, plugging it into (1), and cancelling out 2^{2e} gives $X^2 + Y^2 + Z^2 + W^2 = 2^{2e}XYZW$. If we had $e > 1$, we would get $X^2 + Y^2 + Z^2 + 1 \equiv 0 \pmod{8}$ which is impossible. Thus $e = 1$. But in the same manner, if one of the variables X, Y, or Z was even, the right hand side $4XYZW$ would be divisible by 8 and we arrive again at the contradiction $X^2 + Y^2 + Z^2 \equiv 7 \pmod{8}$. Hence all components x, y, z, and w are congruent to 2 modulo 4.

Finally, we prove by a variation of Vièta jumping that multiples k of odd primes p cannot be divisive: Assume otherwise and let (a, b, c, d) be a funky quadruple which has maximal sum among all funky quadruples (x, y, z, w) satisfying $p \nmid xyzw$ and $x \geq y \geq z \geq w$. Note that $p \nmid 2^4$, so that there exists at least the funky quadruple $(2, 2, 2, 2)$ having these properties. Then, by Vièta's formulae, $d' := abc - d \geq 4a - d \geq 3a > a \geq b \geq c$ gives rise to another funky quadruple (d', a, b, c), so that by the maximality assumption $p \mid d'abc$. But as $p \nmid abcd$, the only possibility is $p \mid d'$. Then, performing two more jumps gives the funky quadruples (c', d', a, b) with $c' := d'ab - c \geq 4d' - c \geq 3d' > d'$ and (d'', c', a, b) with $d'' := c'ab - d' > c'$. We have $c' = d'ab - c \equiv -c \not\equiv 0 \pmod{p}$ and hence also $d'' \equiv c'ab \not\equiv 0 \pmod{p}$, but then (d'', c', a, b) contradicts the maximality of sum of the funky quadruple (a, b, c, d) because $d'' + c' > d + c$.

Problem 3

Problem 3. Let $ABCD$ be a quadrilateral possessing an incenter I. Points K and L are chosen on the segments IA and IC, respectively, such that $CK \perp ID$ and $AL \perp IB$. Points P and Q are chosen on the segments AD and BC, respectively, such that $\angle AKP = \angle ICD$ and $\angle CLQ = \angle IAB$. Prove that $IP = IQ$.

Solution 1

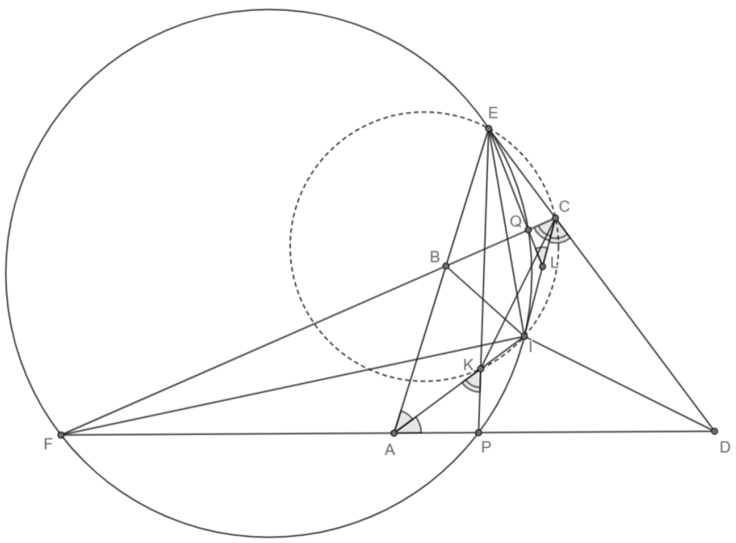


Let $E = AB \cap CD$, $F = BC \cap AD$. From the angle sum of the quadrilateral, we easily obtain that $\angle BIC + \angle AID = 180°$. Clearly, $\angle IKC = 90° - (180° - \angle AID) = \angle AID - 90° = 180° - \angle BIC - 90° = 90° - \angle BIC = \angle ILA$. Notice that $\angle IED = \angle AID - 90° = \angle CKI$, hence the points $E,\ C,\ I,\ K$ lie on the same circle, and since $\angle AKP = \angle ICD$, we conclude that $E \in PK$. Similarly, $E \in QL$. Observe that $\angle EQF = \angle EPF = 180° - \frac{1}{2}(\angle A + \angle C)$. From quadrilateral $IEDF$, we get:

$$\begin{aligned}\angle EIF =& \angle IED + \angle IED + \angle FDE \\ =& (90° - \frac{1}{2}(\angle C + \angle D)) + (90° - \frac{1}{2}(\angle A + \angle D)) + \angle D = 180° - \frac{1}{2}(\angle A + \angle C).\end{aligned}$$

Therefore, points $E,\ Q,\ I,\ P,\ F$ lie on the same circle, and point I is the midpoint of arc PQ, hence $IP = IQ$, which completes the proof.

Solution 2 (More detailed version)



We use directed angles to avoid configuration issues.

Let $E = AB \cap CD$, $F = BC \cap AD$. Then the incircle of $ABCD$ is the incircle the triangles AED and CFD. As P and Q lie on the segments FD and FC respectively, it follows that $\measuredangle PFI = \measuredangle IFQ$. It thus suffices to prove that I, P, Q, F are concyclic. Then it follows that the chords IP and IQ have the same length.

We first prove that P, K and E are collinear, or equivalently, that $\measuredangle IKE = \measuredangle AKP$. As $\measuredangle AKP = \measuredangle ICD$, it is equivalent to prove that I, C, E, K are concyclic. We compute:

$$\begin{aligned}\measuredangle ECK = \measuredangle(ED, CK) &= \measuredangle(ED, ID) + \measuredangle(ID, CK)\\ &= \measuredangle EDI + 90^\circ\\ &= -\measuredangle IAE - \measuredangle AEI = \measuredangle EIA = \measuredangle EIK\end{aligned}$$

Similarly, I, A, E and L are concyclic and L, Q, E are collinear.

We now prove that I, P, E and F are concyclic. We compute:

$$\begin{aligned}\measuredangle PEI = \measuredangle KEI = \measuredangle KCI &= \measuredangle(KC, DI) + \measuredangle(DI, CI)\\ &= 90^\circ + \measuredangle DIC\\ &= 90^\circ - \measuredangle ICD - \measuredangle CDI = \measuredangle DFI = \measuredangle PFI\end{aligned}$$

Similarly, I, Q, E and F are concyclic. Thus I, P, Q, E, F are concyclic.

As $\measuredangle PFI = \measuredangle IFQ$, it follows that $IP = IQ$.

Problem 4

Problem 4. Tom and Jerry play the following game on a 16×16 chessboard. Jerry starts by selecting some, but not all of the squares, and placing one red pebble on each of the selected squares. Tom then places his one black pebble on an empty square and chooses whether he wants to be the hunter or the prey. After that, Tom and Jerry alternate turns with Jerry starting. In each turn, a player chooses one of his pebbles and moves it to a square sharing an edge with the current square. If such a turn would move the pebble outside of the board, the pebble appears on the other side of the board (i.e. if the pebble is on the left border, it appears in the same row on the right border and similarly for the other cases). If at any point in the game a pebble of the hunter is on the same square as a pebble of the prey, the hunter wins and the game ends. An infinite game is a win for the prey. Which of the two players has a winning strategy?

Solution 1

Divide the chessboard into 4×4 squares. Each such square has exactly 4 inner tiles creating a smaller 2×2 square. Jerry places exactly 1 red pebble into each of these squares on one of its inner tiles. In total, he places $\frac{16^2}{4^2} = 16$ pebbles.

Then if Tom places his black pebble and chooses to be the hunter, Jerry will play in such a way that no red pebble leaves its inner 2×2 square. At every moment, black pebble can threaten at most one red pebble, which can then move on at least one safe square. Because Jerry starts, Tom is indeed unable to catch him.

If Tom chooses to be the prey, Jerry will do the following. He will split his red pebbles into two groups of size 8. He will spread these two groups evenly in the first and the last column – thus in each of these columns every second square has a red pebble. Whenever some red pebble can catch the black pebble, he will do it. Otherwise, Jerry will gradually move his first column of pebbles to the second column, then to the third column, etc. Tom's black pebble can't escape the space between these two columns. After enough turns, this space will be small enough so that Tom's black pebble will be caught and Jerry will win.

The described strategy ensures that Jerry will win regardless of Tom's strategy and the proof is therefore completed.

Solution 2 ($5 + 2$ solution for the hunting part)

This is a solution part for Jerry as hunter where we prove that Jerry can win as a hunter with 7 pebbles.

We want to have a stationary vertical wall of 5 red pebbles (the anvil) and a moving vertical wall of two red pebbles (the hammer). We need to prove that the fleeing black pebble cannot pass through the walls after some preparatory moves, and that the hammer will eventually reach the anvil.

Start by moving the red pebbles to a vertical wall in column 4 where the pebbles are in rows 1, 4, 7, 10, 13 and a vertical wall in column 8 where the pebbles are in rows 1 and 9.

Now, if the black pebble is on row 2 to 8, we move the pebble in column 8, row 1 upwards until it either catches up to the row of the black pebble or the black pebble reaches row 9. Similarly, if the black pebble is on row 10 to 16, we move the pebble from row 9 upwards until it either reaches the row of the black pebble or the black pebble reaches row 1.

So we can guarantee a situation where the black pebble shares a row with a red pebble in column 8. Now, if the black pebble is in column 4 after the next move, it can be either captured immediately or it is in row 15. In that case, move the pebble from row 13 up by one, then the black pebble has to leave either to the left or right (or it is immediately captured).

We end up with a starting position where we have 5 red pebbles in column 4 with four holes of size 2 and one hole of size 3, and 2 red pebbles in column 8 where one of the two red pebbles shares a row with the black pebble or is in a row adjacent to the black pebble's row. Because of the toric nature of the board, the black pebble is between the two walls whether it is on the inside or on the "outside".

Now, we play the following strategy: Whenever a red pebble can capture the black pebble, we do that and the game is over, so we can assume that the black pebble will avoid the red-adjacent squares.

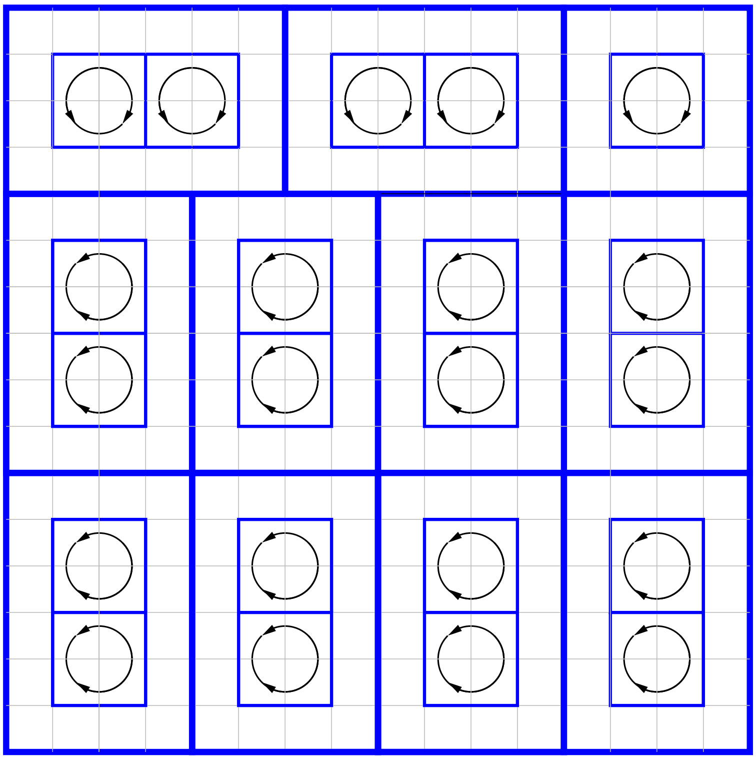
The red pebble that is in the row or adjacent row of the black pebble is the tracker pebble. Whenever the black pebble makes a vertical move that takes it out of the tracker's row and adjacent rows, the tracker makes the same vertical move unless the black pebble moves in the adjacent row of the second red pebble. In that case, the tracking mission switches to the second pebble.

This has the effect that the black pebble can never cross the 2-pebble wall because the tracker pebble is in the way. It is also clear that whenever the black pebble changes direction, the tracker does not need to move. And if the black pebble moves vertically in one direction for long, the tracking mission switch also means that the tracker does not need to move. Therefore, vertical moves will lead to infinitely many "free" moves for the red pebbles which are used to move the two red pebbles (the hammer) alternately horizontally in the direction where the black pebble is closer than the anvil. It is easily seen that it does not harm the mission switch when the two pebbles are not in the same column as the black pebble has to move out of the way or is captured. The only situation where we can run into trouble is if the black pebble is on a square diagonally adjacent to the tracker pebble and we want to move the tracker pebble. In that case, we move the second red hammer pebble so that the vertical interval between the hammer pebbles containing the black pebble gets smaller, so we cannot stay in this situation indefinitely.

The anvil has only exactly one place where a pebble can pass through, the middle of the hole of size 3. Every horizontal move of the black pebble that does not end up on a square adjacent to the hole of size 3 is treated as a free move to move the hammer. If a horizontal move of the black pebble ends up on a square adjacent to the hole of size 3, we make the hole one smaller by moving one of the red pebbles adjacent to the hole vertically. If the black pebble wants to access the new hole of size 3, it needs to make a horizontal move to the right first which gives at least one free move for the hammer until the next attack on the hole. (It also has to make vertical moves which also eventually give free moves to the hammer as we have seen.)

We can conclude that if the black pebble never moves next to a red pebble, it will always stay strictly between the column of the anvil and the column of the hammer whenever the hammer is in a single column. When the anvil and the hammer are in adjacent columns this is not possible anymore so the black pebble must have moved next to a red pebble before that and red wins.

Solution 3 (Sketch: 21 solution for the escaping part)



This is a solution part for Jerry's strategy as prey where we show that Jerry can indefinitely escape with 21 pieces.

As in the first solution, we see that 1 stone can escape indefinitely by just reacting to the black pebble on a 4×4 subboard.

We claim that 2 stones can escape indefinitely by just reacting to the black pebble on a 4×6 subboard. Divide the inner 2×4 board into two halves and assign one pebble to each half where it will always remain. They start out in the outer parts of the 2×4 board.

We will show that the pebbles can not be caught when the black pebble enters the subboard and that they can return to the outer parts of the inner board before the black pebble leaves the subboard.

As long as both red pebbles are on outer squares and none of them is directly threatened, we just move the one of them to its other outer square that is further away from the black pebble.

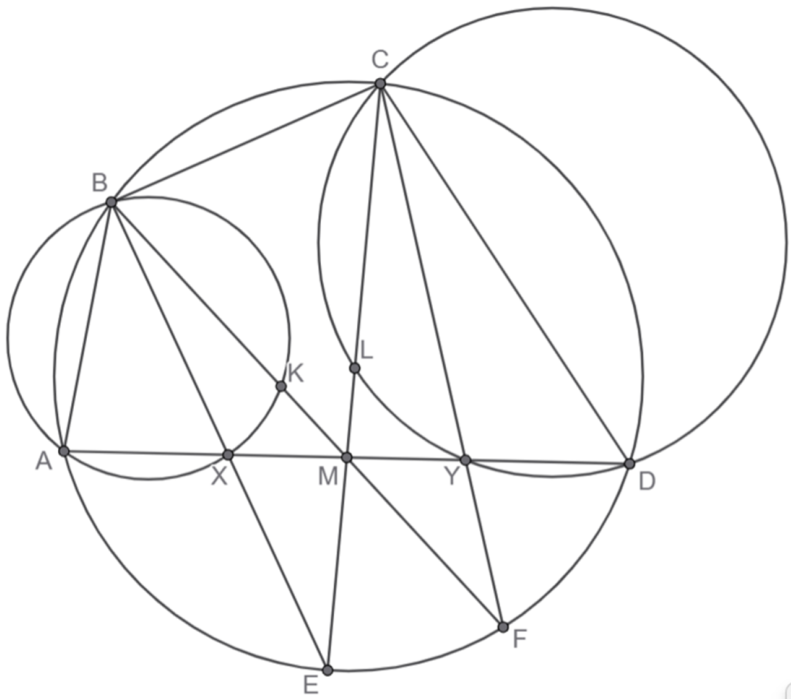
When one of them is threatened and can move to the other outer square, we do so. So, the only situation that forces a red pebble onto an inner square is the black pebble on an outer square of the same 2×2 square. After the red move, the red and black pebbles are always diagonal in the 2×2 square. The black pebble cannot catch the red pebble there, so it will eventually leave the 2×2 square. If it leaves to the outer rim, the red pebble can return to an outer square if necessary (that is possible because they were diagonal before). If the black pebble leaves to the 2×2 square of the other red pebble, it had to be on an inner square before that move, so the previously threatened red pebble is on the diagonally opposite outer field.

We see that each 4×6 is left in the same condition as it was found. So now we can tile the 16×16 board with 10 tiles of size 4×6 and 1 tile of size 4×4. This gives space for 21 pebbles that cannot be captured.

Problem 5

Problem 5. Let M be the midpoint of side AD of a cyclic quadrilateral $ABCD$. Points K and L are chosen on the rays MB and MC respectively such that $\angle DAK + \angle ACM = \angle ABM$ and $\angle LDA + \angle MBD = \angle MCD$. Prove that the points B, C, K, and L lie on one circle.

Solution 1 (1a in the Marking Scheme)



Let the circumcircles of triangles $\triangle AKB$ and $\triangle DLC$ intersect the line AD again at points X and Y respectively. Note that $\angle ABX = \angle ABM - \angle XBM = \angle ABM - \angle KAM = \angle ACM$. Hence the lines BX and CM intersect at a point E lying on the circumcircle of triangle ABC. Similarly, the lines CY and BM intersect at a point F lying on the circumcircle of triangle ABC. For the chords AD, BF and CE, by the Butterfly theorem we obtain $MX = MY$. Therefore $MK \cdot MB = MX \cdot MA = MY \cdot MD = ML \cdot MC$. Hence the points B, C, K, and L lie on one circle.

Solution 2 (1b in the Marking Scheme)

Let the lines BM and CM intersect the circumcircle of triangle ABC again in F and E, respectively. Note that $\angle EBF = \angle EBD - \angle FBD = \angle ECD - \angle FBD = \angle MCD - \angle MBD = \angle LDA$. Similarly, we can show $\angle ECF = \angle DAK$ and we obtain $\angle DAK = \angle EBF = \angle ECF = \angle LDA$. Let X be the intersection of lines AD and BE then $AXKB$ is a cyclic quadrilateral because $\angle XAK = \angle DAK = \angle EBF = \angle XBK$. Similarly, let Y be the intersection of lines AD and CF and we obtain a cyclic quadrilateral $CLYD$. For the chords AD, BF and CE, by the Butterfly theorem we obtain $MX = MY$. Therefore $MK \cdot MB = MX \cdot MA = MY \cdot MD = ML \cdot MC$. Hence the points B, C, K, and L lie on one circle.

Problem 6

Problem 6. Let $n \geq 2$ be an integer and let $a_1, a_2, \dots, a_n$ be non-zero real numbers such that

$$a_1 + \frac{1}{a_2}, \quad a_2 + \frac{1}{a_3}, \quad \dots, \quad a_{n-1} + \frac{1}{a_n}, \quad a_n + \frac{1}{a_1}$$

are all integers. Prove that

$$a_1 a_2 \dots a_n + \frac{1}{a_1 a_2 \dots a_n}$$

is also an integer.

Solution 1

For $n = 2$ we have

$$a_1 a_2 + \frac{1}{a_1 a_2} = \left(a_1 + \frac{1}{a_2}\right)\left(a_2 + \frac{1}{a_1}\right) - 2,$$

so for $n = 2$ the statement is true. Similarly, for $n = 3$ it can be checked that

$$a_1 a_2 a_3 + \frac{1}{a_1 a_2 a_3} = \left(a_1 + \frac{1}{a_2}\right)\left(a_2 + \frac{1}{a_3}\right)\left(a_3 + \frac{1}{a_1}\right) - \left(a_1 + \frac{1}{a_2}\right) - \left(a_2 + \frac{1}{a_3}\right) - \left(a_3 + \frac{1}{a_1}\right).$$

Now generally, fix $n \geq 3$. For $i = 1, \dots, n$ denote $s_i = a_i + 1/a_{i+1}$, where $a_{n+1} = a_1$, and let $P_0 = s_1 s_2 \dots s_n$. We claim that $a_1 a_2 \dots a_n + 1/(a_1 a_2 \dots a_n)$ can be expressed as

$$a_1 a_2 \dots a_n + \frac{1}{a_1 a_2 \dots a_n} = P_0 - P_2 + P_4 - P_6 + \cdots = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \cdot P_{2j},$$

where P_{2j} is defined as follows: Imagine numbers $1, \dots, n$ around a circle and place j non-overlapping dominoes that each cover two consecutive numbers. Then consider the *product* of the numbers s_i, where i is *not* covered. Then P_{2j} is the *sum* of all those products, over all possible placements of j dominoes. For example, for $n = 6$ we claim that

$$a_1 a_2 \dots a_6 + \frac{1}{a_1 a_2 \dots a_6} = s_1 s_2 \dots s_6 - \sum_{cyc}^{6} s_1 s_2 s_3 s_4 + \left(\sum_{cyc}^{6} s_1 s_2 + \sum_{cyc}^{3} s_1 s_4\right) - 2 \cdot 1,$$

because there is a unique way to place no dominoes (leaving each s_i uncovered), there are 6 ways to place 1 domino, $6 + 3$ ways to place 2 dominoes (adjacent or not), and 2 ways to place 3 dominoes (covering everything, so yielding an empty product).

Clearly, each P_j is an integer, so once proved, the claim implies the result. It remains to prove the claim.

Imagine we multiply out each product on the right-hand side, without yet cancelling each resulting expression to lowest terms (that is, we would keep expressions such as $a_1 a_2 / a_1$ etc). Each such expression is of the form

$$t = \frac{\prod_{i \in N} a_i}{\prod_{i \in D} a_i} \cdot \frac{\prod_{i \in I} a_i}{\prod_{i \in I} a_i},$$

where N is a set of indices that appear only in the numerator, D is a set of indices that appear only in the denominator, and I is the set of indices that appear both in the numerator and in the denominator. (We set empty products equal to 1.) Note that the only possible way to get

$i \in I$ is that the product included both s_{i-1} and s_i, and we took $1/a_i$ from $s_{i-1} = a_{i-1} + 1/a_i$ and a_i from $s_i = a_i + 1/a_{i+1}$.

Now group those expressions into batches, where each batch contains the expressions that share the N and D (and differ in I – thus, after cancellation all expressions in one batch yield the same value). Fix a batch and let t_0 be the expression in that batch with $I = \emptyset$. Suppose t_0 comes from a product within P_j. Then any other expression in this batch is obtained by adding some consecutive pairs (s_{i-1}, s_i) to the product (a pair (s_{i-1}, s_i) adds index i to I). To appear on the right-hand side, the product after those pairs (s_{i-1}, s_i) have been added must also be extendable to the full circle on $1, 2, \ldots, n$ by adding dominoes. This allows us to compute the contribution of each batch. We distinguish 3 cases based on the number j of dominoes.

1. $j = 0$: If we took $1/a_{i+1}$ from some s_i and then a_{i+1} from the following s_{i+1} then there would be a cancellation. So the only expressions t_0 without cancellation are precisely the two expressions $a_1 a_2 \ldots a_n$ and $1/(a_1 a_2 \ldots a_n)$, and each of them appears once on the right-hand side, matching the left-hand side.
2. $j < n/2$: Consider the $j > 0$ dominoes that were used to cover parts of the circle, to get the product which contains expression t_0. Each expression in the same batch as t_0 corresponds to a subset of those dominoes. By binomial theorem, the number of odd subsets (of dominoes) is equal to the number of even subsets, so thanks to the alternating signs the total contribution of such a batch to the right-hand side is 0.
3. n is even and $j = n/2$. In this case the expression t_0 is an empty product. There are 2 ways to cover the full circle using dominoes, call them D_1, D_2. Since no domino is used in both D_1 and D_2, each subset of D_1 is different from each subset of D_2, so we can argue for subsets of D_1 and subsets of D_2 separately. Same ss in the previous case, we obtain that the total contribution of such a batch is $2 \cdot 0 = 0$.

Solution 2

Let the given integers be $x_1, x_2, \ldots, x_n$. For each i, we are given

$$a_i + \frac{1}{a_{i+1}} = x_i \quad \Longrightarrow \quad a_i = x_i - \frac{1}{a_{i+1}} = \frac{x_i a_{i+1} - 1}{a_{i+1}}.$$

Notice that a_i is expressed as a rational function of a_{i+1}. Because the sequence is cyclic ($a_{n+1} = a_1$), we can repeatedly substitute this relation to express a_1 as a nested rational function of itself.

To avoid writing out massive fractions, let's define four sequences of numbers: p_k, q_k, r_k, and s_k. We want these sequences to satisfy the relation

$$a_1 = \frac{p_k a_{k+1} + q_k}{r_k a_{k+1} + s_k}.$$

We can build these sequences recursively:

- **Base Case ($k = 0$):** Trivially, $a_1 = \frac{1 \cdot a_1 + 0}{0 \cdot a_1 + 1}$. So we define $p_0 = 1, q_0 = 0, r_0 = 0, s_0 = 1$.
- **Recursive Step:** Suppose $a_1 = \frac{p_k a_{k+1} + q_k}{r_k a_{k+1} + s_k}$. Substituting $a_{k+1} = \frac{x_{k+1} a_{k+2} - 1}{a_{k+2}}$ into this

fraction yields

$$a_1 = \frac{p_k\left(\frac{x_{k+1}a_{k+2}-1}{a_{k+2}}\right) + q_k}{r_k\left(\frac{x_{k+1}a_{k+2}-1}{a_{k+2}}\right) + s_k} = \frac{(p_k x_{k+1} + q_k)a_{k+2} - p_k}{(r_k x_{k+1} + s_k)a_{k+2} - r_k}.$$

This gives us explicit recurrence relations to generate the next coefficients,

$$\begin{aligned} p_{k+1} &= p_k x_{k+1} + q_k \\ q_{k+1} &= -p_k \\ r_{k+1} &= r_k x_{k+1} + s_k \\ s_{k+1} &= -r_k \end{aligned}$$

Since our initial values are integers and $x_i \in \mathbb{Z}$, all terms in these four sequences are strictly integers.

We now prove two properties about these sequences using mathematical induction.

Lemma 1
We claim that for all k, $p_k s_k - q_k r_k = 1$.

- *Proof:* For $k = 0$, $(1)(1) - (0)(0) = 1$. Assume it holds for k. For $k + 1$:

$$\begin{aligned} p_{k+1}s_{k+1} - q_{k+1}r_{k+1} &= (p_k x_{k+1} + q_k)(-r_k) - (-p_k)(r_k x_{k+1} + s_k) \\ &= -p_k r_k x_{k+1} - q_k r_k + p_k r_k x_{k+1} + p_k s_k \\ &= p_k s_k - q_k r_k = 1. \end{aligned}$$

Lemma 2
We claim that the denominator piece relates to the product of our variables: $r_k a_{k+1} + s_k = a_2 a_3 \dots a_{k+1}$.

- *Proof:* For $k = 1$, $r_1 a_2 + s_1 = (r_0 x_1 + s_0)a_2 - r_0 = (0 + 1)a_2 - 0 = a_2$. Assume it holds for k. For $k + 1$ we find

$$\begin{aligned} r_{k+1}a_{k+2} + s_{k+1} &= (r_k x_{k+1} + s_k)a_{k+2} - r_k \\ &= r_k(x_{k+1}a_{k+2} - 1) + s_k a_{k+2}. \end{aligned}$$

  Since $x_{k+1}a_{k+2} - 1 = a_{k+1}a_{k+2}$, substitute this in to get

$$= r_k(a_{k+1}a_{k+2}) + s_k a_{k+2} = (r_k a_{k+1} + s_k)a_{k+2}.$$

  By our inductive assumption, $r_k a_{k+1} + s_k = a_2 \dots a_{k+1}$. Multiplying this by a_{k+2} gives $a_2 \dots a_{k+2}$.

Now, we apply all of this to $k = n$. Let $P = a_1 a_2 \dots a_n$. Because the sequence is cyclic, $a_{n+1} = a_1$.

Using Lemma 2 at $k = n$ yields

$$r_n a_{n+1} + s_n = a_2 a_3 \dots a_{n+1} \quad \Longrightarrow \quad r_n a_1 + s_n = a_2 a_3 \dots a_n a_1 = P.$$

Rearranging gives $r_n a_1 = P - s_n$.

Next, we look back at our main fraction relation at $k = n$,

$$a_1 = \frac{p_n a_1 + q_n}{r_n a_1 + s_n}.$$

Since the denominator is exactly P, we have $a_1 = \frac{p_n a_1 + q_n}{P}$, which rearranges to $(P - p_n)a_1 = q_n$.

We now have a clean system of two equations,

$$a_1 = \frac{P - s_n}{r_n} \tag{2}$$
$$(P - p_n)a_1 = q_n. \tag{3}$$

Substitute equation (1) into equation (2) to eliminate a_1 gives

$$\begin{aligned}(P - p_n)\left(\frac{P - s_n}{r_n}\right) &= q_n \iff \\ (P - p_n)(P - s_n) &= q_n r_n \iff \\ P^2 - (p_n + s_n)P + p_n s_n &= q_n r_n \iff \\ P^2 - (p_n + s_n)P + (p_n s_n - q_n r_n) &= 0.\end{aligned}$$

By Lemma 1, we know that $p_n s_n - q_n r_n = 1$. Plugging this in gives

$$P^2 - (p_n + s_n)P + 1 = 0.$$

Since none of the initial real numbers are zero, $P \neq 0$. We can divide the entire equation by P to arrive at

$$\begin{aligned}P - (p_n + s_n) + \frac{1}{P} &= 0 \iff \\ P + \frac{1}{P} &= p_n + s_n.\end{aligned}$$

Because p_n and s_n are integers built from our sequences, their sum is an integer. Thus, $P + \frac{1}{P} \in \mathbb{Z}$.

Solution 3

Define the integer matrices

$$M_i := \begin{pmatrix} 0 & 1 \\ -1 & a_i + \frac{1}{a_{i+1}} \end{pmatrix}$$

for $i = 1, \ldots, n$ and let $A = M_n \ldots M_1$.

First, note that $\det A = 1$.

Second, note that $\lambda = \frac{1}{a_1 \cdots a_n}$ is an eigenvalue of A. Indeed, one easily checks

$$M_i \begin{pmatrix} 1 \\ \frac{1}{a_i} \end{pmatrix} = \frac{1}{a_i} \begin{pmatrix} 1 \\ \frac{1}{a_{i+1}} \end{pmatrix}$$

for all i, so $A\begin{pmatrix}1\\1/a_1\end{pmatrix}=\lambda\begin{pmatrix}1\\1/a_1\end{pmatrix}$.

Finally, since $\det A = 1$, we have

$$\mathrm{tr}A = \lambda + \frac{1}{\lambda} = a_1\cdots a_n + \frac{1}{a_1\cdots a_n}.$$

Since all entries of A are integers, the trace of A is an integer, which proves the desired claim.