CAPS Match 2024

ISTA, Austria

(First day – July 1, 2024)

1. Determine whether there exist 2024 distinct positive integers satisfying the following: If we consider every possible ratio between two distinct numbers (we include both a/b and b/a), we will obtain numbers with finite decimal expansions (after the decimal point) of mutually distinct non-zero lengths.

   Example: The number 3.14159 has decimal expansion .14159 of length 5.

2. For a positive integer n, an *n-configuration* is a family of sets $\langle A_{i,j} \rangle_{1 \leq i,j \leq n}$. An n-configuration is called *sweet* if for every pair of indices (i,j) with $1 \leq i \leq n-1$ and $1 \leq j \leq n$ we have $A_{i,j} \subseteq A_{i+1,j}$ and $A_{j,i} \subseteq A_{j,i+1}$. Let $f(n,k)$ denote the number of sweet n-configurations such that $A_{n,n} \subseteq \{1, 2, \ldots, k\}$. Determine which number is larger: $f(2024, 2024^2)$ or $f(2024^2, 2024)$.

3. Let ABC be a triangle and D a point on its side BC. Points E, F lie on the lines AB, AC, respectively, such that B lies between A and E and $BE = BD$ and C lies between A and F and $CF = CD$. Let P be a point such that D is the incenter of triangle PEF. Prove that P lies inside the circumcircle Ω of triangle ABC or on it.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

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(Second day – July 2, 2024)

4. Let $ABCD$ be a convex quadrilateral, such that $AB = BC = CD$. There are points X, Y on rays CA, BD, respectively, such that $BX = CY$. Let P, Q, R, S be the midpoints of segments BX, CY, XD, YA, respectively. Prove that points P, Q, R, S lie on a circle.

5. Let $\alpha \neq 0$ be a real number. Determine all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y^2) = f(x - y)f(x + y) + \alpha y f(y)$$

holds for all x, $y \in \mathbb{R}$.

6. Determine whether there exist infinitely many triples (a, b, c) of positive integers such that p divides $\lfloor (a + b\sqrt{2024})^p \rfloor - c$ for every prime p.

Note: $\lfloor x \rfloor$ denotes the largest integer not larger than x.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.