CPS(A) Match 2022

ISTA, Austria

(First day – July 2, 2022)

1. Let $k \leq 2022$ be a positive integer. Alice and Bob play a game on a 2022×2022 board. Initially, all cells are white. Alice starts and the players alternate. In her turn, Alice can either color one white cell in red or pass her turn. In his turn, Bob can either color a $k \times k$ square of white cells in blue or pass his turn. Once both players pass, the game ends and the person who colored more cells wins (a draw can occur).

For each $1 \leq k \leq 2022$, determine which player (if any) has a winning strategy.

2. Find all functions $f\colon (0, \infty) \to (0, \infty)$ such that

$$f\left(f(x) + \frac{y+1}{f(y)}\right) = \frac{1}{f(y)} + x + 1$$

for all $x, y > 0$.

3. Circles Ω_1 and Ω_2 with different radii intersect at two points, denote one of them by P. A variable line ℓ passing through P intersects the arc of Ω_1 which is outside of Ω_2 at X_1, and the arc of Ω_2 which is outside of Ω_1 at X_2. Let R be the point on segment X_1X_2 such that $X_1P = RX_2$. The tangent to Ω_1 through X_1 meets the tangent to Ω_2 through X_2 at T. Prove that line RT is tangent to a fixed circle, independent of the choice of ℓ.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Language: English

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(Second day – July 3, 2022)

4. Given a positive integer n, denote by $\tau(n)$ the number of positive divisors of n, and by $\sigma(n)$ the sum of all positive divisors of n. Find all positive integers n satisfying

$$\sigma(n) = \tau(n) \cdot \left\lceil \sqrt{n} \right\rceil.$$

(Here, $\lceil x \rceil$ denotes the smallest integer not less than x.)

5. Let ABC be a triangle with $AB < AC$ and circumcenter O. The angle bisector of $\angle BAC$ meets the side BC at D. The line through D perpendicular to BC meets the segment AO at X. Furthermore, let Y be the midpoint of segment AD. Prove that points B, C, X, Y are concyclic.

6. Consider 26 letters $A, \ldots, Z$. A *string* is a finite sequence consisting of those letters. We say that a string s is *nice* if it contains each of the 26 letters at least once, and each permutation of letters $A, \ldots, Z$ occurs in s as a subsequence the same number of times. Prove that:

(a) There exists a nice string.

(b) Any nice string contains at least 2022 letters.

(Here, a permutation π of the 26 letters is a *subsequence* of a string s if there exist 26 indices $i_1 < i_2 < \cdots < i_{26}$ such that $\pi = s_{i_1} s_{i_2} \ldots s_{i_{26}}$.)

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Language: English