Czech-Polish-Slovak-Austrian Match 2021

IST Austria (online)

(First day – July 3, 2021)

1. Find all quadruples (a, b, c, d) of positive integers satisfying $\gcd(a, b, c, d) = 1$ and

$$a \mid b + c, \quad b \mid c + d, \quad c \mid d + a, \quad d \mid a + b.$$

2. In an acute triangle ABC, the incircle ω touches BC at D. Let I_a be the excenter of ABC opposite to A, and let M be the midpoint of DI_a. Prove that the circumcircle of triangle BMC is tangent to ω.

3. For any two convex polygons P_1 and P_2 with mutually distinct vertices, denote by $f(P_1, P_2)$ the total number of their vertices that lie on a side of the other polygon. For each positive integer $n \geq 4$, determine

$$\max\{f(P_1, P_2) \mid P_1 \text{ and } P_2 \text{ are convex } n\text{-gons}\}.$$

(We say that a polygon is convex if all its internal angles are strictly less than 180°.)

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Language: English

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(Second day – July 4, 2021)

4. Determine the number of 2021-tuples of positive integers such that the number 3 is an element of the tuple and consecutive elements of the tuple differ by at most 1.

5. The sequence $a_1, a_2, a_3, \ldots$ satisfies $a_1 = 1$, and for all $n \geq 2$, it holds that

$$a_n = \begin{cases} a_{n-1} + 3 & \text{if } n-1 \in \{a_1, a_2, \ldots, , a_{n-1}\}; \\ a_{n-1} + 2 & \text{otherwise.} \end{cases}$$

Prove that for all positive integers n, we have

$$a_n < n \cdot (1 + \sqrt{2}).$$

6. Let ABC be an acute triangle and suppose points $A, A_b, B_a, B, B_c, C_b, C, C_a$, and A_c lie on its perimeter in this order. Let $A_1 \neq A$ be the second intersection point of the circumcircles of triangles AA_bC_a and AA_cB_a. Analogously, $B_1 \neq B$ is the second intersection point of the circumcircles of triangles BB_cA_b and BB_aC_b, and $C_1 \neq C$ is the second intersection point of the circumcircles of triangles CC_aB_c and CC_bA_c. Suppose that the points A_1, B_1, and C_1 are all distinct, lie inside the triangle ABC, and do not lie on a single line. Prove that lines AA_1, BB_1, CC_1, and the circumcircle of triangle $A_1B_1C_1$ all pass through a common point.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

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