

# Czech-Polish-Slovak Match

IST Austria, 23 – 26 June 2019

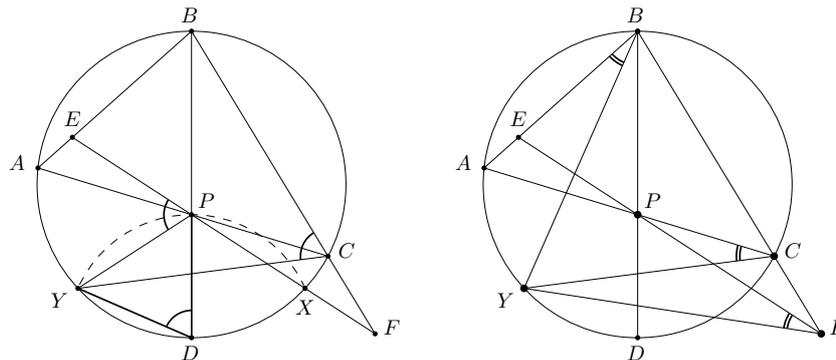
(First day – 24 June 2019)

1. Let  $\omega$  be a circle. Points  $A, B, C, X, D, Y$  lie on  $\omega$  in this order such that  $BD$  is its diameter and  $DX = DY = DP$ , where  $P$  is the intersection of  $AC$  and  $BD$ . Denote by  $E, F$  the intersections of line  $XP$  with lines  $AB, BC$ , respectively. Prove that points  $B, E, F, Y$  lie on a single circle. (Patrik Bak, Slovakia)

**Solution.** First, we show that the quadrilateral  $YPCF$  is cyclic. Indeed, by simple angle-chasing we have

$$\angle YPF = 2 \cdot \angle YPD = 180^\circ - \angle BDY = 180^\circ - \angle BCY = \angle YCF.$$

The rest is angle-chasing again. We have  $\angle EFY = \angle PFY = \angle PCY = \angle ACY = \angle ABY = \angle EBY$  as desired.



2. We consider positive integers  $n$  having at least six positive divisors. Let the positive divisors of  $n$  be arranged in a sequence  $(d_i)_{1 \leq i \leq k}$  with

$$1 = d_1 < d_2 < \dots < d_k = n \quad (k \geq 6).$$

Find all positive integers  $n$  such that

$$n = d_5^2 + d_6^2.$$

(Walther Janous, Austria)

**Solution.** In what follows we shall show that this question has the unique answer  $n = 500$ . Indeed, from  $n = d_5^2 + d_6^2$  we readily infer that  $n$  has to be even. (For, otherwise  $d_5$  and  $d_6$  had to be odd. This in turn would yield  $n$  even.) Therefore  $d_2 = 2$  is fixed. Furthermore from  $d_5 \mid n$  we get  $d_5 \mid d_6^2$  and similarly  $d_6 \mid d_5^2$ . This means:

Every prime dividing  $d_5$  also divides  $d_6$  and vice versa.

If  $d_5$  has only one prime factor, i.e. it is a power of a prime, then  $d_5 = p^k$  and  $d_6 = p^{k+1}$ . But since  $p^k < 2p^k \leq p^{k+1}$ , it follows that  $p = 2$  and  $n = d_5^2 + d_6^2 = 2^{2k} + 2^{2k+2} = 5 \cdot 2^{2k}$ . Therefore either  $n = 20$ , which is not a solution, or

$$d_2 = 2, \quad d_3 = 4, \quad d_4 = 5, \quad d_5 = 8, \quad d_6 = 10,$$

a contradiction.

Now  $d_5$  and  $d_6$  have at least two prime factors  $p$  and  $q$  with  $p < q$  and  $p^2q^2 \mid d_5^2 + d_6^2 = n$ . Then  $d_5 \geq pq$  and since  $1 < p < q, p^2 < pq$  we also have  $d_5 \leq pq$ . Now

$$d_2 = p = 2, \quad \{d_3, d_4\} = \{q, p^2\} = \{q, 4\}, \quad d_5 = pq = 2q, \quad d_6 = p^2q = 4q.$$

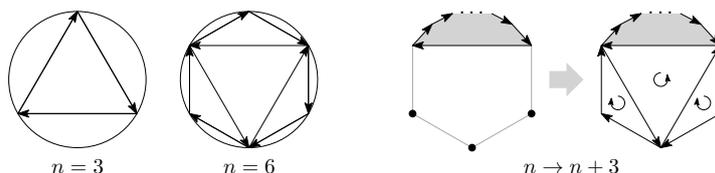
We get  $n = d_5^2 + d_6^2 = 20q^2$ , hence  $q \leq 5$ . Checking the cases  $q = 3$  and  $q = 5$  gives the unique solution  $n = 500$ .

**3.** A dissection of a convex polygon into finitely many triangles by segments is called a *trilateration* if no three vertices of the created triangles lie on a single line (vertices of some triangles might lie inside the polygon). We say that a trilateration is *good* if its segments can be replaced with one-way arrows in such a way that the arrows along every triangle of the trilateration form a cycle and the arrows along the whole convex polygon also form a cycle. Find all  $n \geq 3$  such that the regular  $n$ -gon has a good trilateration. (Josef Greillhuber, Austria)

**Solution.** We show that the regular  $n$ -gon has a good trilateration if and only if  $3 \mid n$ .

Given a regular  $n$ -gon and its good trilateration, color the triangles whose arrows go clockwise in black and the other ones in white. In this way, any two triangles sharing an edge have received different colors and all the triangles sharing an edge with the perimeter of the whole  $n$ -gon have received the same color (wlog black). We say that a segment in the trilateration is *interior* if it is not one of the sides of the  $n$ -gon. Let  $x$  be the number of interior segments. Since each interior segment is a side of precisely one white triangle and the sides of white triangle are all different interior segments, we have  $3 \mid x$ . Arguing likewise for the black triangles, we obtain  $3 \mid x + n$ . Hence  $3 \mid n$ .

It remains to show that when  $3 \mid n$  then the regular  $n$ -gon has a good trilateration. This is straightforward by mathematical induction.



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(Second day – 25 June 2019)

4. Let  $\alpha$  be a given real number. Determine all pairs  $(f, g)$  of functions  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  satisfying

$$xf(x+y) + \alpha \cdot yf(x-y) = g(x) + g(y)$$

for all  $x, y \in \mathbb{R}$ .

(Walther Janous, Austria)

**Solution.** Depending on  $\alpha$ , the solutions are given by:

- If  $\alpha = 1$ , then  $f(x) = C$  and  $g(x) = Cx$  for  $x \in \mathbb{R}$  and  $C$  an arbitrary real constant.
- If  $\alpha = -1$ , then  $f(x) = Cx$  and  $g(x) = Cx^2$  for  $x \in \mathbb{R}$  and  $C$  an arbitrary real constant.
- Else,  $f(x) = g(x) = 0$  for  $x \in \mathbb{R}$ .

Letting  $x = y = 0$ , we obtain  $2g(0) = 0$ , thus  $g(0) = 0$ . Letting  $y = 0$ , we obtain  $xf(x) = g(x)$  for all  $x \in \mathbb{R}$ . Thus, the equation can be rewritten as

$$xf(x+y) + \alpha yf(x-y) = xf(x) + yf(y). \quad (1)$$

Letting  $x = 0$  in (1), we obtain  $\alpha yf(-y) = yf(y)$ . This yields

$$\forall x \neq 0: f(-x) = \alpha f(x). \quad (2)$$

If  $f(x) = 0$  for all  $x \neq 0$ , we let  $y = -x \neq 0$  in (1) and obtain  $xf(0) = 0$ , therefore  $f$  is the zero function, which always solves the equation.

Assume now that there exists  $r \neq 0$  with  $f(r) \neq 0$ . Then it follows from (2) that  $f(r) = \alpha f(-r) = \alpha^2 f(r)$ , thus  $\alpha^2 = 1$  and hence  $\alpha \in \{\pm 1\}$ .

The right-hand side of (1) is symmetric in  $x$  and  $y$ . By switching  $x$  and  $y$ , we thus obtain the equation

$$xf(x+y) + \alpha yf(x-y) = yf(x+y) + \alpha xf(y-x).$$

For  $r \in \mathbb{R}$  we let  $x = (r+1)/2$  and  $y = (r-1)/2$ , which yields

$$f(r) = \alpha \frac{r+1}{2} f(-1) - \alpha \frac{r-1}{2} f(1).$$

By (2), we obtain

$$f(r) = \frac{\alpha f(1)}{2} (\alpha(r+1) - (r-1)).$$

In the case  $\alpha = 1$  this means  $f(r) = f(1)$  for all  $r \in \mathbb{R}$ . In the case  $\alpha = -1$  this means  $f(r) = rf(1)$  for all  $r \in \mathbb{R}$ . Both functions solve the equation, as can be checked easily.

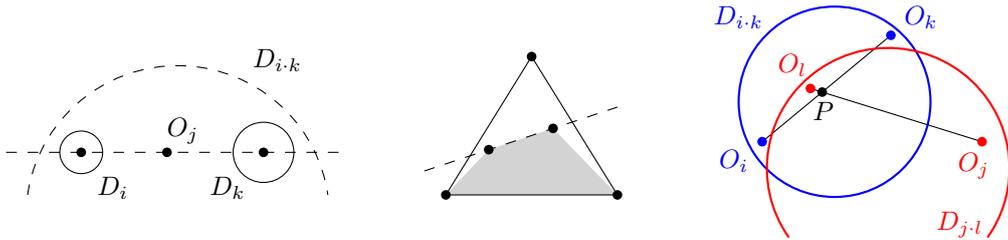
5. Determine whether there exist 100 disks  $D_2, D_3, \dots, D_{101}$  in the plane such that the following conditions hold for all pairs  $(a, b)$  of indices satisfying  $2 \leq a < b \leq 101$ :

1. If  $a \mid b$  then  $D_a$  is contained in  $D_b$ .
2. If  $\text{GCD}(a, b) = 1$  then  $D_a$  and  $D_b$  are disjoint.

(A disk  $D(O, r)$  is a set of points in the plane whose distance to a given point  $O$  is at most a given positive real number  $r$ .) (Josef Greilhuber & Josef Tkadlec, Austria)

**Solution.** Such disks do not exist. Suppose otherwise and denote by  $O_i$  the center of the disk  $D_i$ . Consider the set  $S = \{O_2, O_3, O_5, O_7, O_{11}\}$  of centers of five disks with pairwise coprime indices. We distinguish two cases:

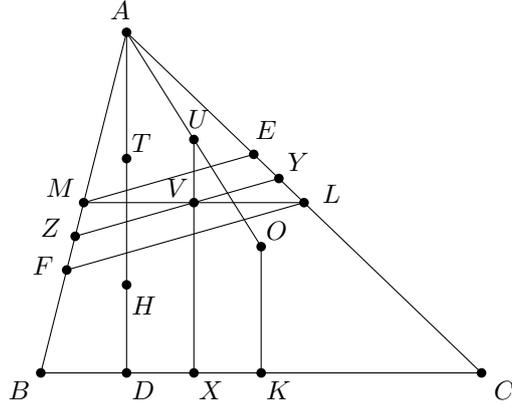
- (i) Some three points from  $S$  lie on a single line: Suppose the three collinear points are  $O_i, O_j, O_k$  in this order. Then  $i \cdot k \leq 7 \cdot 11 \leq 101$ , hence the disk  $D_{i \cdot k}$  is defined. By 1., it contains both  $D_i$  and  $D_k$ , thus it contains  $O_i$  and  $O_k$  and by convexity it also contains  $O_j$ . Therefore, disks  $D_j, D_{i \cdot k}$  intersect, a contradiction with 2.



- (ii) No three points from  $S$  lie on a single line: Then there exist four points from  $S$  that form a convex quadrilateral. (Indeed, either the convex hull of  $S$  contains at least four points, or it is a triangle. In the latter case, the line passing through the two interior points intersects two sides of the triangle and the two interior points form a convex quadrilateral with the endpoints of the side that is not intersected.) Suppose the four vertices of the convex quadrilateral are  $O_i, O_j, O_k, O_l$  in this order. Then, as before, both  $i \cdot k$  and  $j \cdot l$  are at most  $7 \cdot 11 \leq 101$  hence the disks  $D_{i \cdot k}$  and  $D_{j \cdot l}$  are defined. By 1. and by convexity, they both contain the intersection  $P$  of diagonals of  $O_i O_j O_k O_l$ , which is a contradiction with 2.

6. Let  $ABC$  be an acute triangle with  $AB < AC$  and  $\angle BAC = 60^\circ$ . Denote its altitudes by  $AD, BE, CF$  and its orthocenter by  $H$ . Let  $K, L, M$  be the midpoints of sides  $BC, CA, AB$ , respectively. Prove that the midpoints of segments  $AH, DK, EL, FM$  lie on a single circle. (Dominik Burek, Poland)

**Solution.** Denote the midpoints of  $AH, DK, EL, FM$  by  $T, X, Y, Z$ , respectively. Furthermore, let  $O$  be the circumcenter of triangle  $ABC$  and  $U$  the midpoint of  $AO$



(that is, the circumcenter of triangle  $AML$ ). We will show that  $U$  lies on the circle too.

First, we show that  $TUYZ$  is cyclic. In fact, we show that it is an isosceles trapezoid whose line of symmetry is the angle bisector of  $\angle BAC$ : Since  $\angle BAC = 60^\circ$ , we have  $AE = \frac{1}{2}AB = AM$ , thus  $\triangle AME$  is equilateral and, likewise,  $\triangle AFL$  is equilateral. Since  $Y$  and  $Z$  are the midpoints of lateral sides  $EL$ ,  $MF$  of a trapezoid  $ELFM$ , triangle  $AYZ$  is also equilateral and the perpendicular bisector of  $YZ$  is the angle bisector of  $\angle BAC$ . Regarding  $TU$ , since lines  $AT$  and  $AU$  are isogonal in  $\angle BAC$  and  $AF = AL$ , the right triangles  $AFH$  and  $ALO$  are congruent. Thus the perpendicular bisector of  $TU$  is the angle bisector of  $\angle BAC$  as well.

Second, we show that  $UYXZ$  is cyclic: Let  $V$  be the center of parallelogram  $AMKL$ . Since  $V$  is the midpoint of  $ML$ , it lies on the midline  $YZ$  of trapezoid  $MELF$ . Since it is the midpoint of  $AK$ , it also lies on the midline  $UX$  of trapezoid  $AOKD$ . Thus, it remains to check that  $VY \cdot VZ = VU \cdot VX$ , which is straightforward. For the left-hand side, we have  $VY = \frac{1}{2}ME = \frac{1}{4}AB$  and  $VZ = \frac{1}{2}LF = \frac{1}{2}AF$ . For the right-hand side, we have  $VU = \frac{1}{2}OK = \frac{1}{4}AH$  and  $VX = \frac{1}{2}AD$ . Plugging this in, we need  $AB \cdot AF = AH \cdot AD$  which follows from  $BFHD$  being cyclic.

Since both  $TUYZ$  and  $UYXZ$  are cyclic, so is  $TYXZ$ .