Czech-Polish-Slovak Match

IST Austria, 24 – 27 June 2018

(First day – 25 June 2018)

1. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f(x^2 + xy) = f(x)f(y) + yf(x) + xf(x + y).$$

2. Let ABC be an acute scalene triangle. Let D and E be points on the sides AB and AC, respectively, such that $BD = CE$. Denote by O_1 and O_2 the circumcentres of the triangles ABE and ACD, respectively. Prove that the circumcircles of the triangles ABC, ADE and AO_1O_2 have a common point different from A.

3. There are 2018 players sitting around a round table. At the beginning of the game we arbitrarily deal all the cards from a deck of K cards to the players (some players may receive no cards). In each turn we choose a player who draws one card from each of the two neighbours. It is only allowed to choose a player whose each neighbour holds a nonzero number of cards. The game terminates when there is no such player. Determine the largest possible value of K such that, no matter how we deal the cards and how we choose the players, the game always terminates after a finite number of turns.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Language: English

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(Second day – 26 June 2018)

4. Let ABC be an acute triangle with the perimeter of $2s$. We are given three disjoint circles with disjoint interiors with the centres A, B and C, respectively. Prove that there exists a circle with the radius of s which contains all the three circles.

5. In a rectangle with dimensions 2×3 there is a polyline of length 36. Show that there exists a line parallel to two sides of the rectangle, which intersects the other two sides in their interior points and intersects the polyline in fewer than 10 points.

6. We say that a positive integer n is *fantastic*, if there exist positive rational numbers a and b such that

$$n = a + \frac{1}{a} + b + \frac{1}{b}.$$

(a) Prove that there exist infinitely many prime numbers p such that no multiple of p is fantastic.

(b) Prove that there exist infinitely many prime numbers p such that some multiple of p is fantastic.

Time: 4 hours and 30 minutes
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Language: English